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The development of magnetohydrodynamic flow due to the passage of an electric current past a sphere immersed in a fluid

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The development of magnetohydrodynamic flow due to the passage of a uniform electric current past a sphere immersed in an incompressible viscous conducting fluid extending to infinity is considered. The flow field is the response of the fluid to the Lorentz force set up by the electric current and the associated magnetic field. The solution is based on the assumption that the flow field is weak and has a negligible effect on the electromagnetic variables. We also assume that the convection terms in the momentum equation are negligible. The solution, obtained by means of Laplace transforms, is analytic except for the evaluation of an integral which is done numerically. It is shown that the flow field spreads radially from the sphere into the fluid. The rate of development of the flow field increases with the ratio of the conductivity of the sphere to that of the fluid.

1. Introduction

When a conducting fluid extending to infinity is permeated by an electric current which is dependent on the spatial variables a magnetic field is set up. In three-dimensional configurations the associated Lorentz force is rotational and cannot be balanced by a hydrostatic pressure. Thus a flow field is set up. Such flow fields occur in the case of an electric current discharge (Sozou & English 1972) or when a uniform electric current through a conducting fluid is distorted because different fluid regions have different electrical conductivities. Chow & Halat (1969), for example, considered the flow field set up about a solid sphere immersed in a viscous conducting fluid occupying the whole of space and subjected to a uniform electric field. The more general case of the flow field about a spheroid having its axis parallel to the direction of the undisturbed current at infinity was considered by Sozou (1970*a*, *b*).

The work of Sozou & English is an exact solution of the equations of motion coupled with Maxwell's equations. The work of Chow & Halat (1969) and Sozou (1970*a*, *b*) is based on the assumption that the flow field has a negligible effect on the electromagnetic field and, as a first approximation, the inertia terms in the momentum equation can be ignored. The above papers dealt with the steady state which is achieved some time after the application of the electric field and the passage of the electric current. Here we investigate the development of the flow field considered by Chow & Halat. We make the same basic approximations, namely, neglect the effect of the flow field on the electromagnetic field and the convection terms in the momentum equation. We also assume that the electromagnetic field is set up instantaneously, that is, we investigate the response of the fluid to the application of a Lorentz force which is independent of time. The analysis for the development of the flow field about a spheroid in a conducting fluid, with its axis parallel to the direction of the applied electric current, is rather involved. It contains infinite series of spheroidal wave functions and we decided not to pursue that case.

2. Equations of the problem

We consider an infinite incompressible viscous conducting fluid containing an axially symmetric conducting body. This configuration is suddenly subjected to a uniform electric field **E** parallel to the axis of the body. The induced current is **J** and its value at infinity is $\mathbf{J}_0$. We assume that **J** and the associated magnetic field **B** are set up instantaneously. The fluid is then set in motion by the rotational Lorentz force. If we also assume that the fluid velocity **v** is small and its effect on the electromagnetic fields is negligible, that is that **J** is driven solely by the electric field, **B** and **J** satisfy the equations

$$\nabla \times \mathbf{B} = \mu_e \mathbf{J}, \tag{1}$$

$$\nabla \times \mathbf{J} = 0, \tag{2}$$

where μ_e is the magnetic permeability of the medium.

If we assume that the induced flow field **v** is sufficiently weak for the convection terms to be negligible, the momentum equation is

$$\rho\, \partial \mathbf{v}/\partial t + \nabla p + \mu_0 \nabla \times \nabla \times \mathbf{v} - \mathbf{J} \times \mathbf{B} = 0, \tag{3}$$

where ρ, p and μ_0 denote the fluid density, pressure and coefficient of viscosity respectively.

We use cylindrical polar co-ordinates (ϖ, θ, x) with the x axis along the axis of symmetry of the body. Owing to the geometry of the problem **J** and **v** lie in meridian planes through the axis and the magnetic field lines are circles about the axis of symmetry.

It is convenient to express **J** in terms of a current stream function ψ_1 and **v** in terms of a stream function ψ_2, such that

$$\mathbf{J} = \frac{1}{\mu_e \varpi}\left(-\frac{\partial}{\partial x}, 0, \frac{\partial}{\partial \varpi}\right)\psi_1, \tag{4}$$

$$\mathbf{v} = (-\partial/\varpi\, \partial x, 0, \partial/\varpi\, \partial \varpi)\, \psi_2. \tag{5}$$

Using (1) and (4) we find that **B** is given by

$$\mathbf{B} = \hat{\boldsymbol{\theta}} \psi_1(x, \varpi)/\varpi, \tag{6}$$

and hence

$$\mathbf{J} \times \mathbf{B} = -(\psi_1/\mu_e \varpi^2)\, \nabla \psi_1. \tag{7}$$

If we now take the curl of (3) and make use of (5) and (7) we find that ψ_1 and ψ_2 are connected by the equation

$$\frac{2\psi_1}{\mu_0 \mu_e \varpi^2}\frac{\partial \psi_1}{\partial x} = \left(\frac{\rho}{\mu_0}\frac{\partial}{\partial t} - D^2\right) D^2 \psi_2, \tag{8}$$

where
$$D^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial \varpi}.$$

For any particular axisymmetric configuration we must first find ψ_1, that is, solve the equation
$$D^2\psi_1 = 0 \tag{9}$$
and then solve (8) for ψ_2. Equation (9) must be solved for the fluid and the body regions. At the body–fluid interface the normal component of $\mathbf{J}$ and the tangential component of $\mathbf{E}$ $(= \mathbf{J}/\sigma$, σ being the electrical conductivity) are continuous. Having obtained ψ_1 we can then take the Laplace transform of (8) with respect to t and obtain an equation in $\bar{\psi}_2$ and then invert $\bar{\psi}_2$ and find $\psi_2(\varpi, x, t)$.

3. Flow set up around a sphere

For this case it is convenient to use a spherical polar co-ordinate system (r, θ, ϕ) with the origin at the centre of the sphere and the axis $\theta = 0$ along the direction of the undisturbed electric field $\mathbf{E}$. It can be found from books on electromagnetism or easily be shown that for this case
$$\psi_1 = \tfrac{1}{2}\mu_e J_0(r^2 - 2R_0 a^3/r)\sin^2\theta, \tag{10}$$
where a is the radius of the sphere and $R_0 = (\sigma - \sigma_0)/(2\sigma + \sigma_0)$, σ being the conductivity of the fluid and σ_0 that of the sphere. Now in spherical polar co-ordinates (8) becomes
$$\left(\frac{\rho}{\mu_0}\frac{\partial}{\partial t} - D^2\right) D^2\psi_2 = \frac{3\mu_e R_0 J_0^2 a^3}{\mu_0}\frac{a^3}{r^2}\left(1 - 2R_0\frac{a^3}{r^3}\right)\sin^2\theta\cos\theta, \tag{11}$$
where now
$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2}$$
and $\mu = \cos\theta$. If we now set
$$r = aR, \quad t = (a^2\rho/\mu_0)\,T \tag{12}$$
and
$$\psi_2 = f(R, T)\,\mu(1-\mu^2), \tag{13}$$
(11) reduces to
$$\left(\frac{\partial^2}{\partial R^2} - \frac{6}{R^2}\right)\left(\frac{\partial^2}{\partial R^2} - \frac{6}{R^2} - \frac{\partial}{\partial T}\right) f = -\lambda R_0 \frac{1}{R^2}\left(1 - \frac{2R_0}{R^3}\right), \tag{14}$$
where $\lambda = 3\mu_e a^5 J_0^2/\mu_0$.

The boundary conditions are that the velocity vanishes on the sphere, that is,
$$f(1, T) = \partial f(1, T)/\partial R = 0, \tag{15}$$
and the velocity is finite at infinity. This latter somewhat unrealistic condition is derived from the principle of minimum singularity. The reason for the non-vanishing of the velocity field at infinity is due to the neglect of the convection terms in the momentum equation (Sozou 1970*a*).

If we multiply (14) by e^{-sT}, integrate from $T = 0$ to $T = \infty$ and assume that at $T = 0$ the velocity field is zero we obtain
$$\left(\frac{d^2}{dR^2} - \frac{6}{R^2}\right)\left(\frac{d^2}{dR^2} - \frac{6}{R^2} - s\right)\bar{f} = -\frac{\lambda R_0}{s}\frac{1}{R^2}\left(1 - 2\frac{R_0}{R^3}\right), \tag{16}$$

where
$$\bar{f} = \int_0^\infty e^{-sT} f(R,T)\, dT. \tag{17}$$

The solution of (16), making the velocity field finite at infinity, is

$$\bar{f} = \frac{A}{R^2} + \frac{\lambda R_0}{6\alpha^2} R^{\frac{1}{2}} K_{\frac{5}{2}}(\alpha R) \int_d^R \frac{dx}{x K_{\frac{5}{2}}^2(\alpha x)} \int_\infty^x y^{\frac{1}{2}} \left(1 + \frac{2R_0}{y^3}\right) K_{\frac{5}{2}}(\alpha y)\, dy, \tag{18}$$

where A and d are constants, determined from (15), $K_{\frac{5}{2}}$ is the usual Bessel function, that is,

$$K_{\frac{5}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-x} \left(1 + \frac{3}{x} + \frac{3}{x^2}\right), \tag{19}$$

and
$$\alpha = s^{\frac{1}{2}}. \tag{20}$$

On using the Laplace transform of the boundary conditions (15), after some algebra, (18) reduces to

$$\begin{aligned} \frac{6}{\lambda R_0} \bar{f} = &- \frac{1}{\alpha^3 K_{\frac{5}{2}}(\alpha)} \left[\frac{1}{R^2} - \frac{R^{\frac{1}{2}} K_{\frac{5}{2}}(\alpha R)}{K_{\frac{5}{2}}(\alpha)}\right] \int_\infty^1 x^{\frac{1}{2}} \left(1 + \frac{2R_0}{x^3}\right) K_{\frac{5}{2}}(\alpha x)\, dx \\ &+ \frac{R^{\frac{1}{2}} K_{\frac{5}{2}}(\alpha R)}{\alpha^2} \int_1^R \frac{dx}{x K_{\frac{5}{2}}^2(\alpha x)} \int_\infty^x y^{\frac{1}{2}} \left(1 + \frac{2R_0}{y^3}\right) K_{\frac{5}{2}}(\alpha y)\, dy. \end{aligned} \tag{21}$$

In the usual way
$$f(R,T) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{f}(R,s)\, e^{sT}\, ds. \tag{22}$$

It is obvious, from (21), that the origin of the s plane is a branch point of $\bar{f}$. We make $\bar{f}$ single-valued by cutting the s plane all along the negative real axis. Then it is easy to show that $\bar{f}$ has no singularities except for a simple pole at the origin. For small s

$$\frac{6\bar{f}}{\lambda R_0} = -\frac{1}{4s}\left(R^2 - 2 - R_0 + \frac{2R_0}{R} + \frac{1-R_0}{R^2}\right) + O\left(\frac{1}{s^{\frac{1}{2}}}\right). \tag{23}$$

In the usual way the integral on the right-hand side of (22) is reduced to an integral all along the cut of the s plane plus $2\pi i$ times the residue of the pole at the origin. The integral along the cut is from $s = -\infty$ to $s = 0$ on the lower part of the cut, where $s^{\frac{1}{2}} = -i|s^{\frac{1}{2}}|$, and from $s = 0$ to $s = -\infty$ on the upper part, where $s^{\frac{1}{2}} = i|s^{\frac{1}{2}}|$. After some manipulation we obtain

$$f = -\frac{\lambda R_0}{24}\left(R^2 - 2 - R_0 + \frac{2R_0}{R} + \frac{1-R_0}{R^2}\right) - \frac{\lambda R_0}{6\pi R^2} \mathscr{I} \int_0^\infty e^{-pT} F(p,R)\, dp, \tag{24}$$

where $\mathscr{I}$ stands for imaginary part and

$$\begin{aligned} F(p,R) = &\left[1 - \frac{3 + 3ip^{\frac{1}{2}}R - pR^2}{3 + 3ip^{\frac{1}{2}} - p} e^{-ip^{\frac{1}{2}}(R-1)}\right] \frac{1}{p^2(1+ip^{\frac{1}{2}})} \\ &\times [ip^{\frac{1}{2}} + 3 + R_0\{\tfrac{3}{2} + \tfrac{3}{2}ip^{\frac{1}{2}} - \tfrac{1}{4}p + \tfrac{1}{4}ip^{\frac{3}{2}} + \tfrac{1}{4}p^2 e^{ip^{\frac{1}{2}}} E_1(ip^{\frac{1}{2}})\}] \\ &+ \frac{1}{p}(3 + 3ip^{\frac{1}{2}}R - pR^2)\, e^{-ip^{\frac{1}{2}}R} \int_1^R \frac{e^{ip^{\frac{1}{2}}t}}{(3 + 3ip^{\frac{1}{2}}t - pt^2)^2} \\ &\times [ip^{\frac{1}{2}}t^4 + 3t^3 + R_0\{\tfrac{3}{2} + \tfrac{3}{2}ip^{\frac{1}{2}}t - \tfrac{1}{4}pt^2 + \tfrac{1}{4}ip^{\frac{3}{2}}t^3 + \tfrac{1}{4}p^2t^4 e^{ip^{\frac{1}{2}}t} E_1(ip^{\frac{1}{2}}t)\}]\, dt. \end{aligned} \tag{25}$$

$E_1(z)$ represents the exponential integral, that is,

$$E_1(z) = \int_1^\infty \frac{e^{-uz}}{u}\, du = \int_z^\infty \frac{e^{-v}}{v}\, dv.$$

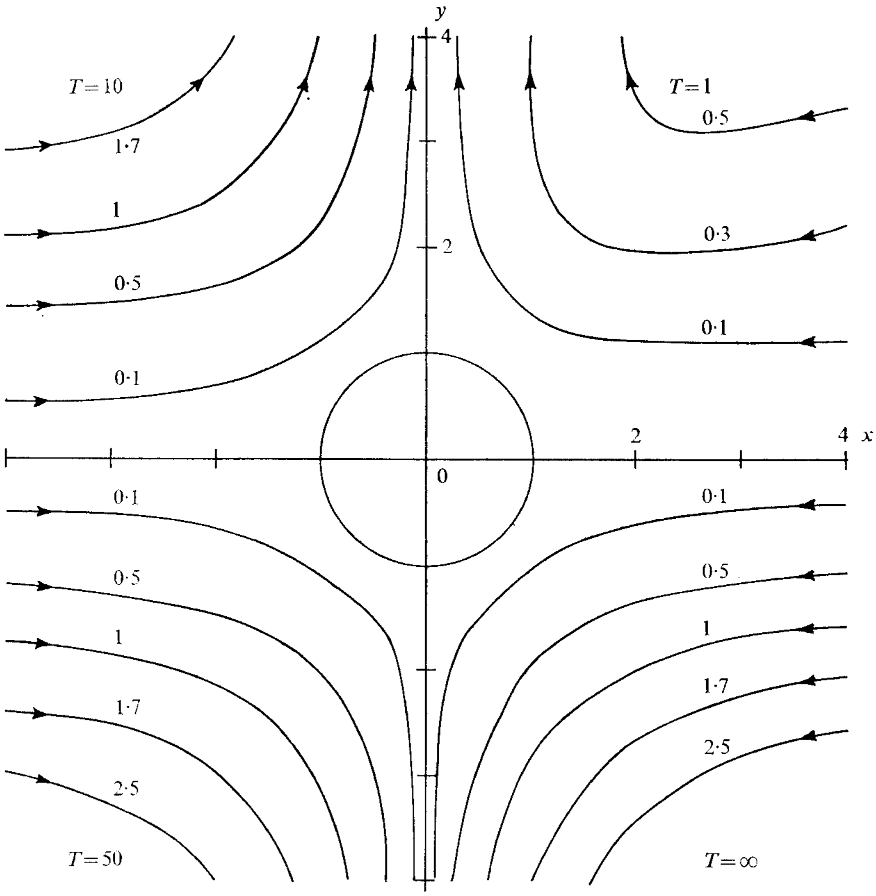

FIGURE 1. Streamlines of the flow field set up by an electric current past an insulating sphere. At infinity the current is parallel to the x axis. The numbers on the curves are values of $24|\psi|/\lambda$. The curves in each quarter refer to the time T shown there.

$R\backslash T$	1	10	50	∞
1·2	0·029	0·048	0·054	0·060
2	0·425	0·797	0·939	1·062
3	0·970	2·344	2·924	3·444
4	1·343	4·196	5·598	6·891
5	1·563	6·137	8·802	11·36
6	1·690	8·017	12·41	16·84

TABLE 1. Values of $-24f/\lambda$ for some R and T for the case of an insulating sphere

The computation of the part of (24) involving integration does not present problems. For small p

$$\mathscr{I}F(p,R) \approx -(2R^5-5R^2+3)/15p^{\frac{1}{2}}. \tag{26}$$

In the range $p = 0$–$0{\cdot}01$, $F(p,R)$ was approximated by (26) and the integral was evaluated analytically by using the first few terms of the expansion of

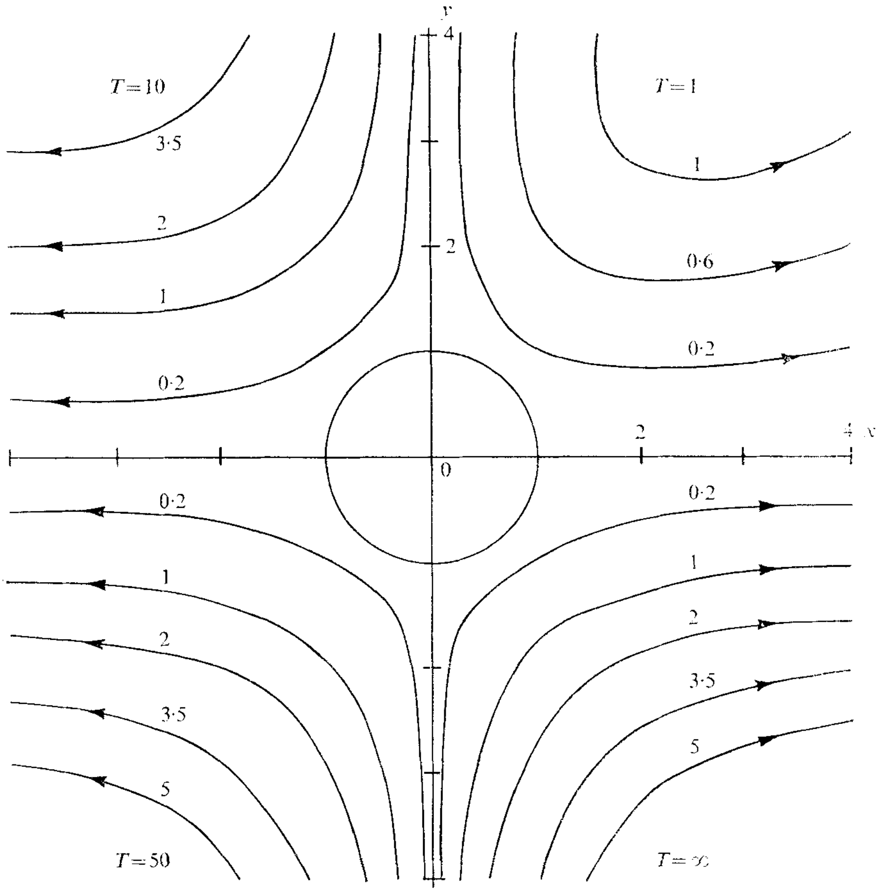

FIGURE 2. Streamlines of the flow field set up by an electric current past a perfectly conducting sphere. At infinity the current is parallel to the x axis. The numbers on the curves are values of $24|\psi|/\lambda$. The curves in each quarter refer to the flow field at the time T shown there.

$R\backslash T$	1	10	50	∞
1·2	0·093	0·135	0·150	0·162
2	1·142	1·962	2·251	2·500
3	2·330	5·326	6·513	7·556
4	3·029	9·170	12·04	14·62
5	3·391	13·11	18·55	23·68
6	3·583	16·88	25·84	34·72

TABLE 2. Values of $24f/\lambda$ for some R and T for the case of a perfectly conducting sphere

$\exp(-pT)$. Thence the integral was evaluated numerically by using Simpson's rule. The integration with respect to p was terminated at $p = 25$. For $p > 25$, $F(p, R)$ is small. The accuracy of our computations was checked by comparing our $f(R, 0)$ with $f(R, \infty)$. For $R > 1{\cdot}4$ our $f(R, 0)$ differed from zero by less than one per cent of $f(R, \infty)$.

At $T = 0$, $f = 0$. As T increases the time-dependent part of f decreases and a flow pattern, which is the response of the fluid to the Lorentz force, is set up. The flow patterns set up are symmetric with respect to the axes $\theta = 0$, π and $\theta = \pm\frac{1}{2}\pi$. For large T

$$f \approx -\frac{\lambda R_0}{24}\left(R^2 - 2 - R_0 + \frac{2R_0}{R} + \frac{1-R_0}{R^2}\right),$$

which is the steady-state solution (Chow & Halat 1969). R_0 varies between -1, corresponding to a perfectly conducting sphere, and 0·5, corresponding to the case where the sphere is an insulator. Figures 1 and 2 show flow patterns for the cases $R_0 = 0{\cdot}5$ and $R_0 = -1$, respectively, and various values of T. In view of (12) the time t taken for the establishment of a certain flow pattern is proportional to $a^2\rho/\mu_0$.

Tables 1 and 2 show values of f/λ for some R and T for the cases of an insulating and a perfectly conducting sphere, respectively. The data for the tables take account of the fact that, in (24) at $T = 0$, the integral of $\mathscr{I}F(p, R)$ differs from the expected value

$$g = -\tfrac{1}{4}\pi R^2[R^2 - 2 - R_0 + 2R_0/R + (1-R_0)/R^2]$$

by $\epsilon(R)$. If the integral of $\mathscr{I}F(p, R)\exp(-pT)$ was found to be $g_1(R, T)$ we assumed that it differed from its expected value by $\epsilon g_1/g$ and modified the data accordingly. This modification affected only the last significant figure of $24f/\lambda$ in the tables. From the tables it can be seen that, for $T = 1$ and $T = 10$, $\partial f/\partial R$ reaches a maximum, from which it decreases. It is, indeed, probable that this is the case for all finite T and the value of R that makes $\partial f/\partial R$ maximum increases with T. Thus the maximum of $\partial f/\partial R$ at $T = 50$ occurs at $R > 6$ and is not shown in the tables. $\partial f/\partial R$ may vanish and f have a maximum in which case the flow field set up is in the form of four eddies about for stagnation points of the R, θ plane. If the maximum value of f, when $T = T_1$, occurs at $R = R_1$, then the points

$$[R_1, \cos^{-1}(\pm 1/3^{\frac{1}{2}})]$$

are stagnation points. As T_1 increases so does R_1 and the stagnation points eventually recede to infinity. This picture, suggesting that the flow field starts from the sphere and spreads outwards, is not unrealistic since this flow field is induced by the presence of the sphere. Also from the tables and the figures it is easy to see that, at any particular finite time T, the intensity of the flow field is closer to that of its steady value as we approach the sphere. Inspection of tables 1 and 2 shows also that for a perfectly conducting sphere the development of the flow field is faster than for an insulating sphere. It appears, therefore, that the smaller (algebraically) the parameter R_0 the faster the development of the flow field.

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